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Gödel's incompleteness theorems

In the twentieth century, mostly because of the "different classes of infinity" problem introduced by George Cantor (1845-1918), a "crisis" about the foundations of mathematics has started... Mathematicians then understood that mathematics should be, just like the legal system of a country, based on some "constitution". David Hilbert's (1862-1943) dream was indeed to create a perfect axiomatic system for mathematics, or at least for domains like geometry or arithmetic, thanks to which all possible statements would be proved to be either true or false. Kurt Gödel (1906-1978) shows, with his incompleteness theorems, that that dream is indeed to remain a dream; and that the Entscheidungsproblem (the German word for "decision problem") cannot be algorithmically solved in arithmetic, therefore that a "Universal Truth Machine" cannot be created...

The first example we can give about this theorem will be simple; it will be about a community where we have two kinds of people: liars and people who tell the truth. Liars always tell lies and people who tell the truth always tell the truth... Now, first problem: let's take a person who always tells the truth. This person will say "I am telling the truth". This part is easy. Now, the bad part: a liar should say "I am a liar" but since that person is a liar she/he will have to lie, and will therefore say "I am telling the truth". As a conclusion, we cannot tell, just by asking her/him, if that person is a liar or not. But, the problem is indeed worse: remember that our aim in creating perfect axiomatic systems was that for any given correct sentence, we can say (not say, demonstrate based on the axioms) that it either is true or false. Let's take the sentence "I am a liar": based on the things a person telling the truth and a liar says I've explained a few lines ago, no one can actually pronounce that sentence! And, here is the problem announced by Gödel: we have a sentence about which we cannot say anything, so we cannot prove that it is true, but we cannot prove that it is false either, even though the sentence is valid!

Variants of the same example (which was "the liar paradox") also exist: we have the barber paradox, where we have a town with only one barber (who's male), and where any person who does not shave himself is shaved by the barber and any person who shaves himself is not shaved by the barber... Problem: what does the barber do? He actually cannot either shave or not shave himself, therefore our town cannot exist. We can see a simple way of completing our system, just by saying that "the barber shaves himself"; but this introduces a deep problem in our system: we have contradiction between our axioms!

The Grelling-Nelson paradox (which is also occasionally called "Weyl's paradox") about autological words (words that define themselves, such as "short" or "pentasyllabic") and heterological words (words that do not define themselves, such as "long" or "monosyllabic"): is the word "heterological" autological? If it is then it isn't, and if it isn't then it is. Russel's paradox is also a good one: let's suppose the set of all sets that do not contain themselves. In Cantor's system, this is a well-defined set, since we can say that $M = \{A \mid A \notin A\}$ to define it as M. Does this list contain itself? If it does, then it should be a member of itself, so it should not be in that list. If it doesn't, then it should be contained in that list, which is itself. As you probably have remarked by yourself, the "barber paradox" is indeed an applied version of Russel's paradox.

Another important thing Gödel says is called the second incompleteness theorem, which states that "no consistent system can be used to prove its own consistency". This result was even more devastating for mathematicians that were trying to construct a philosophical approach to mathematics, known as Hilbert's program: David Hilbert proposed that the consistency of more complicated systems, such as real analysis, could be proved in terms of simpler systems. Therefore, the consistency of all of (complicated) mathematics could be reduced to basic (simpler) arithmetic. Gödel's second incompleteness theorem shows that basic arithmetic cannot even be used to prove its own consistency, so it certainly cannot be used to prove the consistency of anything stronger. (at this point one may seriously wonder the friendship level between Hilbert and Gödel)

Gödel's incompleteness theorems are very "dangerous" (yes, mostly to Hilbert's ideas), which makes them interesting, and therefore popular... but also misinterpreted!

First of all, the theorem does not say that every axiomatic system is necessarily incomplete: for example, absolute geometry with the following rules is incomplete:

1. Any two points can be joined by a straight line.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.

Still, if we add to it a fifth rule saying that *if two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough* (which is also the "axiom of parallels"), we indeed obtain a complete system, and this system is called Euclidian geometry...

Another example of complete system is the Presburger arithmetic, which is a first order theory of the natural numbers with addition... Even though it is not as powerful as the Peano axioms because it does not have multiplication, it is quite interesting since Mojzesz Presburger proved in 1929 that there is an algorithm which decides for any given statement in Presburger arithmetic whether it is true or not (on the other side, concepts such as divisibility or prime number cannot be proved in Presburger arithmetic... We could therefore see the Presburger arithmetic a system that's simpler than the Peano arithmetic but that has the advantage of being complete). Later on, in 1974, Fischer and Rabin even proved that every algorithm which decides the truth of Presburger statements has a runtime of at least $2^{(2^{(cn)})}$ for some constant c and where n is the length of the Presburger statement. The problem therefore needs more than polynomial run time (indeed, even more than exponential time!), but returns you either "yes" or "no"...

The first theorem says that *in any consistent formalization of mathematics that is sufficiently strong to axiomatize the natural numbers one can construct a true statement that can be neither proved nor disproved within that system itself*. The theorem therefore only applies to systems that allow you to define the natural numbers as a set (it is not sufficient for the system to contain the natural numbers): you must be able to express the concept "x is a natural number" using your axioms and first-order logic. There actually are plenty of systems that contain the natural numbers and are complete. For example, both the real numbers and complex numbers have complete axiomatizations.

What are the consequences of those theorems for us, us being computer scientists?

In computability theory the halting problem is a decision problem which can be informally stated as follows: *given a description of an algorithm and its initial input, determine whether the algorithm, when executed on this input, ever halts (completes), the alternative being that it runs forever without halting*. Now, let's ask the question Gödel has asked for him incompleteness theorems: what happens if we apply this program to itself? Alan Turing proved in 1936 that a general algorithm to solve the halting problem for all possible inputs cannot exist, and this result was later generalized in the field of recursive functions to Rice's theorem which shows that all non-trivial decision problems are undecidable in a system that is Turing-complete. So, the conclusion is immediate: no unique algorithm can tell you, just by looking at a piece of code, if that piece of code will halt on all inputs or for some input will just loop forever...

Gödel's second incompleteness theorem is also very important, mostly if we rephrase it like that: *if an axiomatic system can be proved to be consistent from within itself, then it is inconsistent*. Therefore, if we have a program that would automatically verify for us the programs we write, and when we ask this problem "are you true?", and the programs tells us "yes, I am", then it means that it indeed is not! This paradox is, in my opinion, powerful enough to make most programmers working in those domains go really mad at you when talked about it...

I will also talk a bit about the consequences it has on the ultimate computer project, artificial intelligence... Arguments against strong AI based on some philosophical consequences derived from an interpretation of Gödel's proof have been around for many years since their initial formulation by Lucas (in 1961) and their recent revival by Penrose (in 1989 and later on in 1994), and the main argument runs as follows: *Gödel's incompleteness theorem shows that computational systems are limited in a way that humans are not. In any consistent formal system powerful enough to do a certain sort of arithmetic there will be a true sentence — a Gödel sentence (G) — that the system cannot prove. Nonetheless, we can see that the Gödel sentence is true, thus we have a capacity that the formal system lacks. G also stands for a number — a Gödel number Gp — which results from the assignment of a code number to each sentence in the language of P that expresses metamathematical sentences. So, Gödel's incompleteness theorem is proved by finding a sentence Gp which is not provable from P*. Mainly, the biggest discussions are about the mind being a non-deterministic (therefore non-deductive) system whereas a computer, which is based on mathematical realities, is therefore a deterministic (deductive) and is limited by the set of axioms it will run on...

Still, people like Chalmers or McCullough think that the human mind is also tightly related with Gödel's theorems, and say that Gödel argument did not prove that human reasoning had to be non computable: it only proved that if human reasoning was computable, then it had to either be unsound, or it had to be inherently impossible for a human to know both what a human's own reasoning powers were and to also know that they were sound. Therefore, they think that the only reason why we're not able of formalizing our reasoning system is that we are not conscious enough about it.

As a conclusion, we can say that Gödel's incompleteness theorems had a big impact on the construction of modern mathematics and currently have a big impact on the way we see programming and artificial intelligence projects... A browsing on the internet will also show that discussions linking theological issues and Gödel's incompleteness theorems exist. The theorems are, at first sight, "weird enough to be false", still become interesting after some thinking...

References

Alan Turing, On computable numbers, with an application to the Entscheidungsproblem, Proceedings of the London Mathematical Society, Series 2, 42 (1936), pp 230–265

K. Gödel: Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I. (http://home.ddc.net/ygg/etext/godel/)

Karlis Podnieks, Around Goedel's Theorem (http://www.ltn.lv/~podnieks/gt.html)

Hilbert's second problem (http://aleph0.clarku.edu/~djoyce/hilbert/problems.html#prob2)

Joao Teixeira (Universidade Federal de S. Carlos, Brazil) - Computational Complexity and Philosophical Dualism (http://www.bu.edu/wcp/Papers/Cogn/CognTeix.htm)

Fatih Gelgi, Artificial Intelligence vs. the Mind (http://www.fountainmagazine.com/articles.php?SIN=a22010e74c&k=226&832044153&show=part1)

J. McCarthy, Awareness and Understanding in Computer Programs (http://psyche.cs.monash.edu.au/v2/psyche-2-11-mccarthy.html)

D. McCullough, Can Humans Escape Gödel? (http://psyche.cs.monash.edu.au/v2/psyche-2-04-mccullough.html)

The Wikipedia online encyclopedia, http://www.wikipedia.org